Week 8 Description
11.1 Sequences
11.2 Series
There is a lot more to sequences, including more definitions.

Definition: A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$ that is,
$a_1 < a_2 < a_3 < \ldots$.
It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$
A sequence is **monotonic** if it is either increasing or decreasing.

Definition A sequence (a_n) is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{ for all } n \geq 1$$

It is bounded below if there is a number m such that

$$m \geq a_n$$

for all $n \geq 1$

If it is bounded above and below, then $\{a_n\}$ is a **bounded** sequence.
Monotone Convergence Theorem Every bounded monotonic sequence converges.

There is a proof in the book. It casually drops the "Completeness Axiom" on you as if you have heard of it before and it is no big deal. Actually it is a big deal. If you want, check out the **axioms for real numbers** (you might want to look up "axiom" first) and see how mundane most of them are. Then look up why it is called the "Completeness" Axiom

One fact good to know is that if $\lim\limits_{n\to\infty} a_n = L$ then $\lim\limits_{n\to\infty} a_{n-1} = L$ as well. You can work from the definition, but the sequence is the sequence.

For example if $\lim\limits_{n\to\infty} a_n = \dfrac{2}{3}$ Then $\lim\limits_{n\to\infty} 3 + \dfrac{1}{a_{n-1}} = 3 + \dfrac{3}{2}$

Two examples follow

1. Find the limit of the sequence given by the recursion

$$a_1 = 1, a_n = \frac{1}{4 - a_{n-1}}$$

The fact that this is decreasing bounded below by 0 insures that is has a limit, but proving it is decreasing requires induction. If you know mathematical induction, try it.

Using the fact that if we call the limit L then both $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_{n-1} = L$ we need only solve

$$L = \frac{1}{4 - L}$$

a fairly easy quadratic which gives

$$L = 2 \pm \sqrt{3}$$

but there is only one limit. Writing out the first few terms shows $a_n < 2$ making the answer

$$\lim_{n\to\infty} a_n = 2 - \sqrt{3}$$

2. Let

$$a_1 = a_2 = 1, a_n = 2a_{n-1} + 3a_{n-2}$$

First few terms are

$$1, 1, 5, 13, 41, 121, ...$$

an unbounded increasing sequence.

Put $b_n = \dfrac{a_n}{a_{n-1}}$ and find $\lim_{n\to\infty} b_n$ The first few terms are

$$1, 5, \frac{13}{5}, \frac{41}{13}, \frac{121}{41}, ...$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \frac{2a_{n-1} + 3a_{n+2}}{a_{n-1}} = 2 + 3\frac{a_{n-2}}{a_{n-1}} = 2 + \frac{3}{b_{n-1}}$$

You should supply reasons for each equal sign, especially the last one. But now we are almost done as before. If we call the limit L we solve

$$L = 2 + \frac{3}{L}$$

or

$$L^2 - 2L - 3 = 0 \iff (L-3)(L+1) = 0$$

and so $L = 3$

11.2 section on Series

It is extremely important for everything that follows to really understand sigma notation so you can read the text. We will be concerned mainly with whether a series converges or not. Sometimes we can find the sum of the series, most of the time we cannot.

A series converges if the sequence of partial sums converges. Written out longhand, if the series is

$$a_1 + a_2 + a_3 + a_4 + ...$$

then the partial sums are

$$s_1 = a_1, s_2 = a_1 + a_2, s_3 = a_1 + a_2 + a_3, s_4 = a_1 + a_2 + a_3 + a_4$$

and so on. Already you can see why we need sigma notatio

Definition: Given a series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots$$

let s_n denote its nth partial sum:

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{k=1}^{\infty} a_k$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{k=1}^{\infty} a_n = s$$

The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

One series whose convergence we know and whose sum we know is the geometric series

$$a + ar + ar^2 + ar^3 + ar^4 + ... = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

If $|r| < 1$ then

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

and if $|r| \geq 1$ it diverges.

3. For example

$$\sum_{n=1}^{\infty} 2 \times \left(\frac{3}{4}\right)^{n-1} = \frac{2}{1-\frac{3}{4}} = \frac{2}{\frac{1}{4}} = 4 \times 2 = 8$$

4. To find the sum

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$$

Our old friend partial fractions gives

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$

It is illuminating to write out the first few terms without doing the arithmetic to see what happens to this "telescoping sum"

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2} = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} \dots$$

The pattern being clear and the only two numbers that survive are $1 + \frac{1}{2} = \frac{3}{2}$

To be more precise the partial sums will be

$$s_n = \frac{3}{2} - \frac{1}{n+2}$$

and

$$\lim_{n\to\infty} \frac{3}{2} - \frac{1}{n+2} = \frac{3}{2}$$

Theorem: If the series $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$ in other words if the sum converges the terms must get closer and closer to zero.

The converse is not true. In other words, just because the terms of a series approach zero in the limit, it does not mean the sum converges.

5. Worlds most famous example of a series where the terms go to zero but the sum does **not** converge is the "harmonic series"

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

The reason this series diverges is worked out in the text, and also later proved by the integral test.